The Use of Non-Euclidean Geometry in Measurements of Periodically Loaded Transmission Lines*

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Summary—The propagation characteristics of periodically loaded transmission lines can be deduced from impedance measurements taken with a series of different terminating configurations in a manner analogous to the "nodal shift" method of measuring microwave junction characteristics. The non-Euclidean properties of impedance transformations form a particularly simple approach for analyzing measurements in the case of the loaded line.

I. Introduction

A NUMBER of techniques used in testing the disk-loaded guide for the large Stanford linear electron accelerator[1] are not only of interest to workers with similar structures, but also provide an interesting example of the use of the non-Euclidean properties of impedance or reflection coefficient charts which have been described by Deschamps.[2]

The basic property to be used in this discussion is that one can define a non-Euclidean geometry on the set of reflection coefficients or impedances, such that transformation through a nondissipative two-terminal-pair network constitutes a rigid displacement in this geometry. This concept is of general use in network problems, as illustrated in footnote reference 2. The ideas presented are valid for the interpretation of measurements made at a single frequency. No attempt is made to discuss frequency dependence. We shall plot our diagrams on the conventional Smith chart. A few properties of this representation are listed below. Let words in *italics* represent the description in the non-Euclidean system. Deschamps uses the adjective "hyperbolic" to denote quantities in this system. *Straight lines* are circles orthogonal to the edge of the chart. *Circles* are circles but with the *center* displaced from the Euclidean center. The *unit of length* is proportional to the logarithm of the voltage standing-wave ratio in the following sense: the *distance* between the center of the Smith chart and a given point is half the logarithm of the vswr (voltage standing-wave ratio) at the point in question; the *distance* between *two points* is the value that would be obtained by applying any *rigid displacement* which moves one of the points to the center of the chart. *Angles* between *intersecting straight lines* are equal to the angles between tangents at the point of intersection. For a description of additional properties of this geometry, the reader is referred to the references. The treatment used in this discussion is, from personal preference, entirely geometric. Use is freely made of the theorems of Euclid, care being taken not to use any that depend on the parallel axiom.

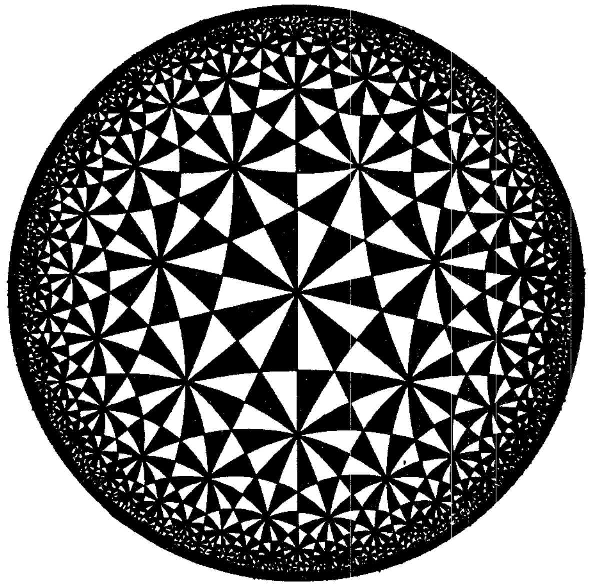

Fig. 1—Regular mosaic in a non-Euclidean geometry.

Many *ruler-and-compass* constructions carry over directly to the non-Euclidean system if practical methods are available for drawing a *straight line* through *two points*, and for drawing a *circle* with a given *center*. Deschamps[2] gives a useful method for the former which involves the "projective-chart" as well as the Smith-chart representation. In the work described in the present contribution, the lines of constant reactance on the printed Smith-chart were used as a set of French curves for drawing *straight lines* in conjunction with a tracing table. To draw *circles* use is made of the definition of *distance*. If a *circle* is to have its *center* at $(\text{vswr})_0$ and the *radius* is to be $\frac{1}{2}$ ln $(\text{vswr})_R$, then the points on the *circle* will have vswr values lying between $(\text{vswr})_0 \times (\text{vswr})_R$ and $(\text{vswr})_0 \div (\text{vswr})_R$. The extremes will lie on a Euclidean diameter of the Smith chart drawn through the desired *center*. The two extremes also lie on a Euclidean diameter of the desired circle which may now be drawn at once. In order to give a graphic illustration of this geometry, Fig. 1 shows a mosaic pattern of congruent triangles and regular heptagons.

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[1] M. Chodorow, E. L. Ginzton, W. W. Hansen, R. L. Kyhl, R. B. Neal, and W. K. H. Panofsky, "Stanford high-energy linear electron accelerator (Mark III)," *Rev. Sci. Instr.*, vol. 26, p. 134; February, 1955.

[2] G. A. Deschamps, "Application of non-Euclidean geometry to the analysis of waveguide junctions," (summary) PROC. IRE, vol. 40, p. 743; June, 1952; also, *J. Appl. Phys.*, vol. 24, p. 1046; August, 1953; and "A hyperbolic protractor," Fed. Telecom. Labs., Nutley, N. J.; 1953. A more elegant and sophisticated treatment is given in the Proc. of the Symposium on Modern Network Synthesis, vol. 1, p 277, Polytech. Inst. of Brooklyn.

II. Measurements on Periodically Loaded Lines

Fig. 2 shows a typical experimental situation. The coupling network is some device for connecting the loaded line to an ordinary transmission line. The measuring device is usually a probe and slotted section for measuring standing-wave ratio in a transmission line, but this is not essential to the argument; the same considerations would apply to a completely lumped network. The plunger can be used for terminating the loaded line in various impedances, either purely reactive or partly resistive, and generally unknown.

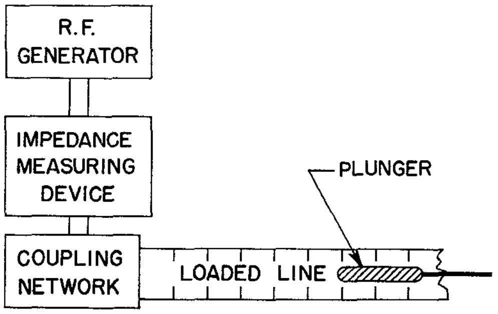

Fig. 2—A diagram of a typical experimental setup.

Since it is possible to generate in periodic structures waves which may be defined as "pure traveling waves," the entire structure of analysis developed for conventional transmission lines can be taken over directly to systems containing periodically loaded lines or combinations of these with ordinary lines, provided only that some care is taken in the interpretations.

The purpose of an experiment may be to measure the propagation constant of the loaded line, to check its uniformity, or to measure its characteristic impedance as seen through the coupling network. The concept of impedance and reflection coefficient in periodically loaded transmission lines has been discussed by Slater[3] and by Jaynes.[4]

III. Geometric Analysis

The *rigid displacement* corresponding to the transformation through one period (one disk) of the loaded line will have one fixed point at the characteristic impedance Z_0 of the structure. Within the pass band of the structure the characteristic impedance will lie inside the Smith chart (rather than at *infinity*). The only possible *displacement* is then a *rotation* about Z_0. Transformation through several sections must then consist of a succession of *rotations* about the same point. The total *angle of rotation* will be $N\theta$ where θ is the *angle of rotation* for a single period.

[3] J. C. Slater, "Microwave Electronics," D. Van Nostrand and Co., Inc., New York, N. Y., ch. 7; 1950.

[4] E. T. Jaynes, "The concept and measurement of impedance in periodically loaded waveguides," *J. Appl. Phys.*, vol. 23, p. 1077; October, 1952.

If we plot with Z_0 at the center of the Smith chart the *rotation* will appear Euclidean. Consider the experiment consisting in the placing of a purely reactive, totally reflecting termination successively in equivalent positions in cavities $a, b, c, d, \cdots$ of the loaded line. The resulting impedances as seen, for example, in cavity a and plotted on a Smith chart normalized to the characteristic impedance of the loaded line must appear as in Fig. 3 (though it is not yet clear from the discussion whether it is possible to make such measurements). Here ϕ is the phase shift per section in the line. Since the plunger is purely reactive the reflection coefficient has unit amplitude and all points lie on the edge of the chart if there are no losses present. (Loss in the structure will be considered later.) The factor 2 in 2ϕ is the usual one found on Smith-chart plots.

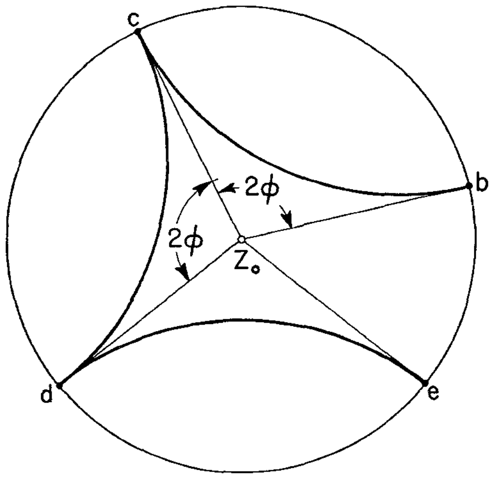

Fig. 3—Unclosed *regular polygon* with center at center of chart, sides of infinite *length*.

We may now ask what will be observed by an actual impedance measurement made through a non-lossy coupling network such as in Fig. 2. The plunger is again to be purely reactive as in the previous paragraph. First suppose the coupling network exactly matches the loaded line to the uniform line. This means by definition that Z_0 of the loaded line falls at the center of the Smith chart. It follows at once that we must again observe exactly Fig. 3 except for an arbitrary *rigid rotation*. In general, however, the coupling will not be matched. The observed points will fall in the manner indicated in Fig. 4. The first step in interpreting the measurements is to determine the position of Z_0. Connect points $b, c, d, e, \cdots$ with *straight lines*. These form a *regular polygon* (*unclosed*) in the sense that all vertices are equivalent and all sides are equivalent. It can be readily seen that it must be a *regular polygon* in Fig. 3. Since Fig. 4 represents a *rigid displacement* of Fig. 3, the figure must still be a *regular polygon*. Ordinarily, location of the center of a regular polygon is an elementary geometric construction. Unfortunately in this case the *vertices* are at *infinity*, the *vertex angles* are vanishingly small, and the sides have *infinite length*.[5] We construct the *straight lines* (cZ_0 and Z_0d) which are the *loci of points equidistant* from two adjacent *sides* of the *polygon*. (It may not be at once clear how to perform such a construction, or even that the locus in question is a *straight line*. Fortunately, a simple construction is available.) This may be done on the Smith chart by superimposing the diagram on the printed Smith chart so that the *vertex* under discussion falls at the point $Z=\infty$. The adjacent *sides* then lie on lines of constant reactance X. The *line* of constant $X=(X_1+X_2)/2$ is the desired *locus*. The proof is left to an appendix.

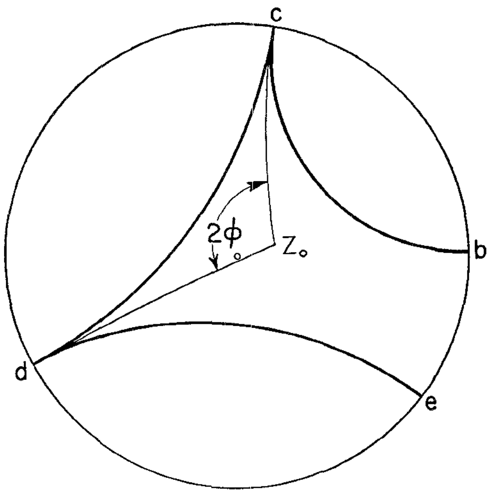

Fig. 4—Unclosed *regular polygon* with center displaced from center of chart. This figure is *congruent* to Fig. 3.

The intersection of all these loci determines the center Z_0 of the polygon. The reflection coefficient of the coupling network can be read off directly from the position of this *center*. Since we have already drawn the *lines* connecting the *center of the polygon* with the vertices, the *angles* 2ϕ can be measured at once as the *angles* formed by these *lines* at the *center of the polygon*.

If it should happen that point f falls upon point b as in Fig. 5, then the *polygon* reduces to a "*square*" and the *center* of the polygon can be determined by drawing the two *diagonals*. This is a special case of the type of nodal-shift[6] measurement suggested by Deschamps[2] for use with uniform lines. It can be used if the loaded line is transmitting in exactly the $\pi/4$ or $3\pi/4$ mode.

The $\pi/2$ mode is commonly used in traveling-wave linear accelerators. The method just described breaks down at or near this point. Fig. 6 shows measurements on a periodically loaded line very near the $\pi/2$ mode. In order to determine the Z_0 point with precision a second set of plunger positions can be measured giving rise to points b', c', d', e', etc.

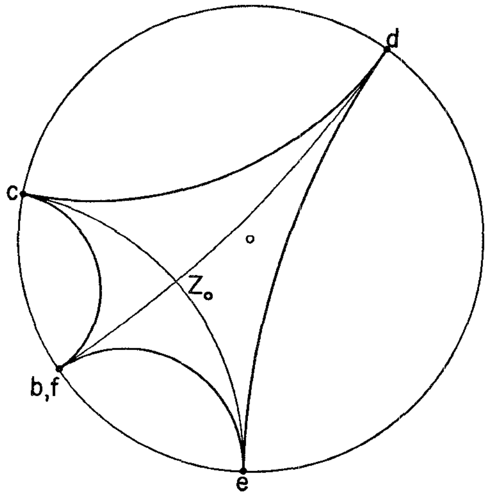

Fig. 5—*Square*, showing method of locating its *center* by drawing the *diagonals*.

These points determine another *polygon* which is also *centered* about Z_0. Unfortunately, in a disk-loaded structure (at least if the coupling holes are small), there is one set of points (say, b, c, d, e) which is insensitive to plunger position and corresponds to successive complete detuning of the set of cavities. All other points (like b', c', d', e') can be obtained only by positioning the plunger in the transition region between successive cavities with great precision. Fig. 6 indicates how the location of Z_0 is determined.

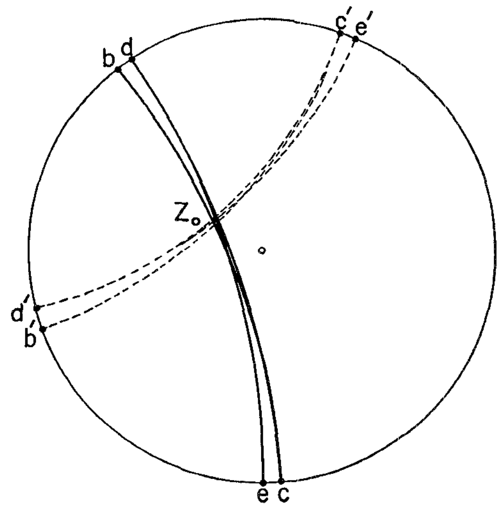

Fig. 6—Typical data obtained in a loaded line operating near the $\pi/2$ mode.

[5] If the reader is disturbed by a *regular polygon* with *vertices at infinity* he may consider this discussion as the limiting case of an experiment in which the plunger has a slight loss.

[6] E. Feenberg, "The relation between nodal positions and standing wave ratio in a complete transmission system," *J. Appl. Phys.*, vol. 17, pp. 530–532; June, 1946 and N. Marcuvitz, "On the representation and measurement of waveguide discontinuities," Proc. IRE, vol. 36, pp. 728–735; June, 1948.

Another method of determining the location of Z_0 which may give greater accuracy and is feasible near the $\pi/2$ mode involves the use of a lossy plunger. The *points* now fall inside the chart and the difficulties with points at *infinity* disappear. The Z_0 *point* may be determined as the intersection of the *perpendicular bisectors of the sides* or *bisectors* of the *vertex angles* as seen in Fig. 7. Ordinary "ruler-and-compass" constructions are suitable.

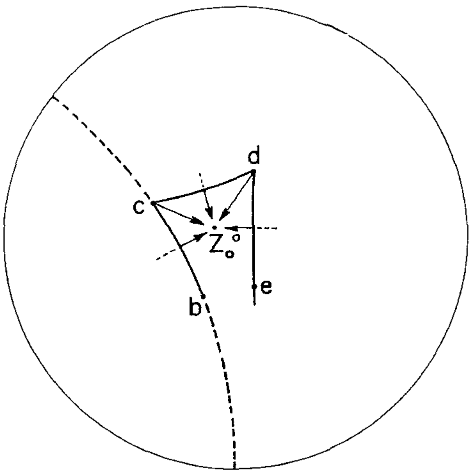

Fig. 7—Unclosed *regular polygon* constructed from data obtained with lossy plunger.

This method does not break down at the $\pi/2$ mode. It is not as accurate a method for measuring phase angles. It has been used at Stanford in conjunction with the totally reflecting plunger.[7] By a little trial and error, a lossy plunger has been manufactured with a vswr of less than 1.1 when placed appropriately in a cavity of the large Stanford linear accelerator. With such a plunger the Z_0 *point* can be located accurately by inspection.

Another method of analyzing lossy plunger data consists of drawing a *circle* through the data points. The points must fall on a *circle* because they form a *regular polygon*. A *circle* is always also a circle in the Euclidean plane so the construction can be made by conventional methods. The *center of the circle* coincides with the *center of the polygon*. Location of the *center of a circle* is one of the simplest constructions on a Smith chart. It is illustrated in Fig. 8. The diameter WXY of the chart through the Euclidean center O of the circle is drawn intersecting the circle at A and B. The *center* O' *of the circle* is given by vswr values from the center of the Smith chart according to the formula

$$\text{vswr}_{O'} = \sqrt{\frac{\text{vswr}_B}{\text{vswr}_A}},$$

or if the circle does not link the origin,

$$\text{vswr}_{O'} = \sqrt{\text{vswr}_B \times \text{vswr}_A}.$$

This is the reverse of the method of drawing a *circle* described in section I.

[7] Chodorow *et al*, *op. cit.*, p. 161.

IV. Reflections in Loaded Lines

A direct extension of the methods described above permits the determination of reflections which may occur within periodic structures. Such discontinuities may be the result of faulty manufacture or may be introduced intentionally. The experimental arrangement will be the same as before. The measurement will now consist of one set of plunger settings on the near side of the point of reflection and another set on the far side. The interpretation of each set will be made as usual; it is then possible by conventional microwave network methods to determine the nature of the impedance transformation which lies between the two sets.

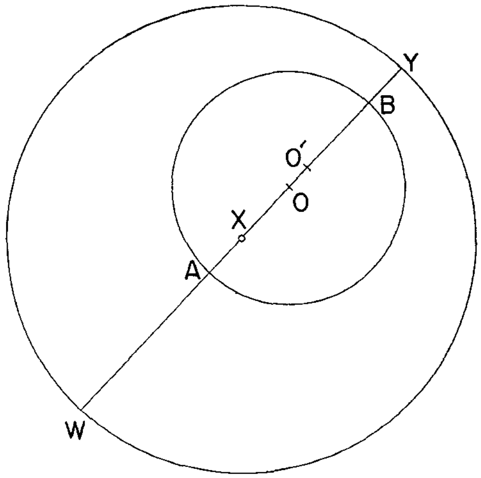

Fig. 8—Illustration of method of locating the *center* of a circle.

V. Lossy Loaded Lines

The impedance transformation through a general lossy network cannot be described as a *rigid displacement* in the non-Euclidean system. (All possible *displacements* already represent nondissipative transformations.) In the lossy case it is found that the Smith chart is mapped into a smaller circular region entirely within the Smith chart (possibly tangent to the boundary). All positions and sizes of the reduced circle are permitted, subject to the above restrictions. If the reduced circle is considered to be a new reduced size representation of the entire *plane*, then the lossy network transformation constitutes a *rigid displacement* within this region.

We shall consider only the case of sufficiently small attenuation constant so that the attenuation through a few cavities may be neglected. We may then assume that the result of an experiment with a reactive plunger as described in section III will be points lying on a circle (rigorously they fall on a spiral).

The entire analysis procedure described for nonlossy lines carried over unaltered to this case if we define the non-Euclidean quantities with respect to the region inside the reduced circle (measured for example with the use of the reactive plunger). The result is that we can establish the characteristic impedance and propagation constant in the region of the reactive plunger, as seen through the lossy network. We have thus taken care of the periodic loading aspect of the problem and have reduced it to the corresponding problem with conventional transmission lines, for which methods of interpretation are available.[2] Fig. 9 shows a sample measurement for this case.

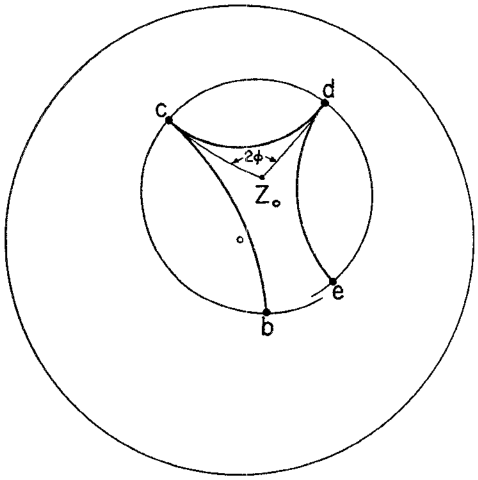

Fig. 9—Unclosed, *regular polygon* constructed from data obtained with a lossless plunger some distance down a slightly lossy transmission line.

VI. Conclusion

We have shown how measurement of the various network properties of transmission line circuits can be extended to circuits containing periodically loaded transmission lines. The use of the non-Euclidean approach permits the analysis to be made with a minimum of mathematical complexity, and provides a convenient conceptual framework.

The methods here described are readily extended but with considerable complexity to the case where attenuation cannot be neglected even between two successive cavities.

Appendix: Location of the Center of a "Polygon" with Vertices at Infinity

It will be recalled that the existence of a *"regular" polygon* was based on physical arguments about the network under examination. These arguments also show that if a transformation is used to move the *center* of the *polygon* to the center of the Smith chart, then the figure will possess rotational symmetry about the center in the laboratory coordinates as well as in the hyperbolic. Under these conditions it is clear at once from symmetry considerations that the *center* of the *polygon* is located at the intersection of *lines* through the vertices (radii of the Smith chart) which have the property of being equidistant from the two adjacent sides of the figure in both the laboratory and the hyperbolic coordinates. The problem is now to obtain a formula for constructing these *lines* even when the *center* of the *polygon* does not happen to be at the center of the Smith chart.

If the *polygon* in question (not necessarily centered on the Smith chart) is superimposed on the printed Smith chart with the vertex in question at $Z=\infty$, then the two adjacent *sides* of the *polygon* will lie along lines of constant reactance jX. This follows at once, since the family of constant reactance lines on the chart are the family of all *straight lines* which can be drawn through the point $Z=\infty$. Call the reactance values X_1 and X_2. Then it is asserted that the curve of constant reactance equal to $(X_1+X_2)/2$ is the desired *line*. To see this let us apply the transformation which consists of adding a series reactance of $-j(X_1+X_2)/2$. This is certainly a permissible transformation since it is physically realizable and is nondissipative. The result is lines of constant reactance $j(X_1-X_2)/2$; 0; $j(X_2-X_1)/2$. The figure is symmetric about the curve $jX=0$, which has all the necessary symmetry properties to be the desired *line* through the center of the polygon. Since this is a property in the hyperbolic geometry it is invariant under transformation and is the desired *line* under all conditions.

The same construction can be applied to the other *vertices* to give two or more *lines* whose intersection defines the *center* of the *polygon*.

VII. Acknowledgment

The assistance rendered by K. B. Mallory and Charles Süsskind of Stanford University in the preparation of this paper is gratefully acknowledged.